APPENDIX G: VECTOR POTENTIAL, FIELD MOMENTUM, AND GAUGE TRANSFORMATIONS

This section is included because it is hard to find the magnetic vector potential $\mathbf{A}$ discussed thoroughly in one place, and we need the vector potential in superconductivity. It may seem mysterious that the hamiltonian of a particle in a magnetic field has the form derived in (18) below:

$$H = \frac{1}{2M}\left(\mathbf{p} - \frac{Q}{c}\mathbf{A}\right)^2 + Q\varphi \ , \tag{1}$$

where Q is the charge; M is the mass; $\mathbf{A}$ is the vector potential; and φ is the electrostatic potential. This expression is valid in classical mechanics and in quantum mechanics. Because the kinetic energy of a particle is not changed by a static magnetic field, it is perhaps unexpected that the vector potential of the magnetic field enters the hamiltonian. As we shall see, the key is the observation that the momentum $\mathbf{p}$ is the sum of two parts, the kinetic momentum

$$\mathbf{p}_{\mathrm{kin}} = M\mathbf{v} \tag{2}$$

which is familiar to us, and the potential momentum or field momentum

$$\mathbf{p}_{\mathrm{field}} = \frac{Q}{c}\mathbf{A} \ . \tag{3}$$

The total momentum is

$$\boxed{\mathbf{p} = \mathbf{p}_{\text{kin}} + \mathbf{p}_{\text{field}} = M\mathbf{v} + \frac{Q}{c}\mathbf{A}\ ,} \tag{4}$$

and the kinetic energy is

$$\frac{1}{2}Mv^2 = \frac{1}{2M}(Mv)^2 = \frac{1}{2M}\left(\mathbf{p} - \frac{Q}{c}\mathbf{A}\right)^2 . \tag{5}$$

The vector potential[1] is related to the magnetic field by

$$\mathbf{B} = \operatorname{curl}\mathbf{A}\ . \tag{6}$$

We assume that we work in nonmagnetic material so that **H** and **B** are treated as identical.

Lagrangian Equations of Motion

To find the Hamiltonian, the prescription of classical mechanics is clear: we must first find the Lagrangian. The Lagrangian in generalized coordinates is

$$L = \frac{1}{2}M\dot{q}^2 - Q\varphi(\mathbf{q}) + \frac{Q}{c}\dot{\mathbf{q}}\cdot\mathbf{A}(\dot{\mathbf{q}})\ . \tag{7}$$

This is correct because it leads to the correct equation of motion of a charge in combined electric and magnetic fields, as we now show.

In Cartesian coordinates the Lagrange equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0\ , \tag{8}$$

and similarly for y and z. From (7) we form

$$\frac{\partial L}{\partial x} = -Q\frac{\partial \varphi}{\partial x} + \frac{Q}{c}\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right)\ ; \tag{9}$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + \frac{Q}{c}A_x\ ; \tag{10}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = M\ddot{x} + \frac{Q}{c}\frac{dA_x}{dt} = M\ddot{x} + \frac{Q}{c}\left(\frac{\partial A_x}{\partial t} + \dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{z}\frac{\partial A_x}{\partial z}\right)\ . \tag{11}$$

Thus (8) becomes

$$M\ddot{x} + Q\frac{\partial \varphi}{\partial x} + \frac{Q}{c}\left[\frac{\partial A_x}{\partial t} + \dot{y}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) + \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] = 0\ , \tag{12}$$

[1]For an elementary treatment of the vector potential see E. M. Purcell, *Electricity and magnetism,* 2nd ed., McGraw-Hill, 1984.

or

$$M\frac{d^2x}{dt^2} = QE_x + \frac{Q}{c}[\mathbf{v}\times\mathbf{B}]_x \ , \tag{13}$$

with

$$E_x = -\frac{\partial\varphi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} \ ; \tag{14}$$

$$\mathbf{B} = \operatorname{curl}\mathbf{A} \ . \tag{15}$$

Equation (13) is the Lorentz force equation. This confirms that (7) is correct. We note in (14) that $\mathbf{E}$ has one contribution from the electrostatic potential φ and another from the time derivative of the magnetic vector potential $\mathbf{A}$.

Derivation of the Hamiltonian

The momentum $\mathbf{p}$ is defined in terms of the Lagrangian as

$$\mathbf{p} \equiv \frac{\partial L}{\partial\dot{\mathbf{q}}} = M\dot{\mathbf{q}} + \frac{Q}{c}\mathbf{A} \ , \tag{16}$$

in agreement with (4). The hamiltonian $H(\mathbf{p},\mathbf{q})$ is defined by

$$H(\mathbf{p},\mathbf{q}) \equiv \mathbf{p}\cdot\dot{\mathbf{q}} - L \ , \tag{17}$$

or

$$H = M\dot{q}^2 + \frac{Q}{c}\dot{\mathbf{q}}\cdot\mathbf{A} - \frac{1}{2}M\dot{q}^2 + Q\varphi - \frac{Q}{c}\dot{\mathbf{q}}\cdot\mathbf{A} = \frac{1}{2M}\left(\mathbf{p} - \frac{Q}{c}\mathbf{A}\right)^2 + Q\varphi \ , \tag{18}$$

as in (1).

Field Momentum

The momentum in the electromagnetic field that accompanies a particle moving in a magnetic field is given by the volume integral of the Poynting vector, so that

$$\mathbf{p}_{\text{field}} = \frac{1}{4\pi c}\int dV\,\mathbf{E}\times\mathbf{B} \ . \tag{19}$$

We work in the nonrelativistic approximation with $v \ll c$, where v is the velocity of the particle. At low values of v/c we consider $\mathbf{B}$ to arise from an external source alone, but $\mathbf{E}$ arises from the charge on the particle. For a charge Q at $\mathbf{r}'$,

$$\mathbf{E} = -\nabla\varphi \ ; \qquad \nabla^2\varphi = -4\pi Q\,\delta(\mathbf{r}-\mathbf{r}') \ . \tag{20}$$

Thus

$$\mathbf{p}_f = -\frac{1}{4\pi c}\int dV\,\nabla\varphi\times\operatorname{curl}\mathbf{A} \ . \tag{21}$$

By a standard vector relation we have

$$\int dV\, \nabla\varphi \times \text{curl}\, \mathbf{A} = -\int dV\, [\mathbf{A} \times \text{curl}\,(\nabla\varphi) - \mathbf{A}\, \text{div}\, \nabla\varphi - (\nabla\varphi)\, \text{div}\, \mathbf{A}] \ . \quad (22)$$

But curl $(\Delta\varphi) = 0$, and we can always choose the gauge such that div $\mathbf{A} = 0$. This is the transverse gauge.

Thus, we have

$$\mathbf{p}_f = -\frac{1}{4\pi c}\int dV\, \mathbf{A}\, \nabla^2\varphi = \frac{1}{c}\int dV\, \mathbf{A} Q\, \delta(\mathbf{r} - \mathbf{r}') = \frac{Q}{c}\mathbf{A} \ . \quad (23)$$

This is the interpretation of the field contribution to the total momentum $\mathbf{p} = M\mathbf{v} + Q\mathbf{A}/c$.